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DENSITY OF CARMICHAEL NUMBERS WITH THREE PRIME FACTORS

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ABSTRACT. We get an upper bound of $O(x^{5/14+o(1)})$ on the number of Carmichael numbers $\leq x$ with exactly three prime factors.

1. INTRODUCTION

A Carmichael number is a composite number n which satisfies the condition $a^n \equiv a \bmod n$ for every integer a. The smallest Carmichael number is 561. The Carmichael numbers have many interesting properties. For example, it is known that they are square-free and the product of at least three primes [5]. The reader may consult [4], [7], [8], [11] for more on Carmichael numbers.

The problem of proving the existence of infinitely many Carmichael numbers was a long-standing open problem until it was solved recently, by Alford, Granville and Pomerance [1]. They also gave a lower bound for the number of Carmichael numbers less than a given number x. Let $C(x)$ denote the number of Carmichael numbers up to x. They showed that $C(x) > x^{2/7}$ for all sufficiently large x.

Let $C_k(x)$ denote the number of Carmichael numbers up to x with k prime factors where $k \geq 3$. It is an open problem to show that the function $C_3(x)$ is unbounded. It is not known whether any of the functions $C_k(x)$ is unbounded. Pomerance et al. [9] proved that $C_3(x) = O(x^{2/3})$. Damgård et al. [3] improved this to $C_3(x) \leq (1/4)x^{1/2}(\log x)^{11/4}$ for all $x \geq 1$. An unpublished estimate of $O(x^{2/5+o(1)})$ for $C_3(x)$ was obtained by S. W. Graham. We show that for sufficiently large x, $C_3(x) = O(x^{5/14+o(1)})$. Granville (see [8]) has conjectured that $C_k(x) = x^{1/k+o_k(x)}$ for $x \to \infty$. Our upper bound for $C_3(x)$ comes very close to his conjectured value.

2. PROOF OF OUR BOUND

We state our result on the upper bound for $C_3(x)$ and give its proof. The proof is very similar to that in Damgård et al. [3].

Theorem 2.1. *Let $C_3(x)$ denote the number of Carmichael numbers up to x with exactly three prime factors. Then, for all sufficiently large x we have $C_3(x) = O(x^{5/14+o(1)})$.*

Proof. If n is a Carmichael number with three prime factors p, q, r with $2 < p < q < r$, then $n-1 \equiv 0 \bmod p-1$, $n-1 \equiv 0 \bmod q-1$, $n-1 \equiv 0 \bmod r-1$.

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Let $g = \gcd(p-1, q-1, r-1)$ and a, b, c be such that $p-1 = ga$, $q-1 = gb$, $r-1 = gc$; then $a < b < c$. The congruences given above imply that $gbc + b + c \equiv 0 \bmod a$, $gac + a + c \equiv 0 \bmod b$ and $gab + a + b \equiv 0 \bmod c$. These three congruences can be replaced by the single congruence $g(ab + ac + bc) + a + b + c \equiv 0 \bmod abc$ by observing that a, b, c are pair-wise coprime. This is true because $\gcd(a, b, c) = 1$ and $c \equiv 0 \bmod \gcd(a, b)$, $b \equiv 0 \bmod \gcd(a, c)$, $a \equiv 0 \bmod \gcd(b, c)$ implies that $\gcd(a, b) = \gcd(a, c) = \gcd(b, c) = 1$. Hence, if a, b, c are given, then g is determined modulo abc.

We count the number N of quadruples (g, a, b, c) which satisfy the above conditions and $g^3abc \le x$. Thus $C_3(x) \le N$. We write $N = N_1 + N_2 + N_3$ where N_1 is the number of quadruples (g, a, b, c) such that $g > abc$, N_2 is the number of quadruples (g, a, b, c) such that $G < g \le abc$ where $G = x^{3/14}$, N_3 is the number of quadruples (g, a, b, c) such that $g \le G$ and $g \le abc$ where G is as above. □

Estimate for N_1

If (a, b, c) are given, then the number of g with $g^3abc \le x$, g in a particular residue class modulo abc and $g > abc$ is at most $(x/abc)^{1/3}/abc$, which is $x^{1/3}/(abc)^{4/3}$. Hence

$$N_1 \le \sum_{a<b<c} \frac{x^{1/3}}{(abc)^{4/3}} < \frac{\zeta^3(4/3)x^{1/3}}{6}$$

where ζ is the Riemann zeta function. Thus $N_1 = O(x^{1/3})$.

Estimate for N_2

For each coprime triple (a, b, c) there is at most one g that satisfies the condition $g(ab + ac + bc) + a + b + c \equiv 0 \bmod abc$ and $g \le abc$. If $g > G$ and $g^3abc \le x$, then $abc \le x/G^3$. Thus N_2 is at most the number of triples (a, b, c) with $a < b < c$ and $abc \le x/G^3$. Hence,

$$\begin{aligned} N_2 &\le \sum_{1\le a<x^{1/3}/G} \quad \sum_{a<b<(x/aG^3)^{1/2}} \quad \sum_{b<c\le x/abG^3} 1 \\ &< \sum_a \sum_b \frac{x}{abG^3} \quad < \quad \sum_a \frac{x}{aG^3} \ln\left(\left(\frac{x}{aG^3}\right)^{1/2}\right) \\ &< \frac{x}{2G^3}\left(1 + \ln\left(\frac{x^{1/3}}{G}\right)\right) \ln\left(\frac{x}{G^3}\right) \quad < \quad \frac{x}{6G^3}(\ln(x))^2 \\ &= O(x^{5/14+o(1)}), \text{ since } G = x^{3/14}. \end{aligned}$$

Thus $N_2 = O(x^{5/14+o(1)})$.

Estimate for N_3

In this case $g \le G$ and $g \le abc$ where $G = x^{3/14}$. Let $g(ab+bc+ac)+a+b+c = \lambda abc$ where $\lambda \ge 1$ is a positive integer. Then $(\lambda a - g)bc = ga(b+c) + a + b + c$. We note that $6gbc \ge g(ab + bc + ac) + a + b + c = \lambda abc$ implies that $\lambda a \le 6g$. We break the range for g, a, b as $G_1 \le g \le 2G_1$, $A \le a \le 2A$, $B \le b \le 2B$. We consider two cases: $B \ge Ax^{1/14}$ and $B < Ax^{1/14}$.

THE CASE $B \geq Ax^{1/14}$

We have,

$$\begin{aligned}
|\lambda a - g| &= \frac{ga(b+c)+a+b+c}{bc} \\
&= ga(1/c+1/b)+a/bc+1/c+1/b \\
&< 2ga/b+3/b \quad (\text{since } 1/c<1/b \text{ and } a<b<c) \\
&= O(G_1A/B) \quad (\text{since } g \leq 2G_1, a \leq 2A, B \leq b) \\
&= O(x^{2/14}) \quad (\text{since } G_1 \leq G = x^{3/14} \text{ and } B \geq Ax^{1/14}).
\end{aligned}$$

We can fix g in $x^{3/14}$ ways since $g \leq G = x^{3/14}$. For a given value of g, λa has only $O(x^{2/14})$ choices since $|\lambda a - g| = O(x^{2/14})$. So we can fix g, a, λ in $O(x^{5/14+o(1)})$ ways. Now b, c have only $x^{o(1)}$ choices since $(g-\lambda a)bc+(b+c)(ga+1)+a=0$ implies $[(g-\lambda a)b+1+ga][(g-\lambda a)c+1+ga] = (1+ga)^2-(g-\lambda a)a$. We must ensure that $ga-\lambda a^2 \neq (ga+1)^2$. It is easily checked that this must be the case by looking, modulo a, at both sides of this inequality.

THE CASE $B < Ax^{1/14}$

Let $AJ \leq B \leq 2AJ$; then $J \leq x^{1/14}$. We consider the equality $g(ab+bc+ca)+a+b+c = \lambda abc$. We fix λ, a, b first and show that g, c have $x^{o(1)}$ choices by considering the equality $gc(a+b)+c(1-\lambda ab)+gab+a+b=0$. This equality implies that $[\lambda ab-1-(a+b)g][ab+(a+b)c] = (\lambda ab-1)ab+(a+b)^2$ which is positive. Thus, for fixed λ, a, b there are $\leq x^{o(1)}$ choices for g, c. Since $\lambda a \leq 6g \leq 12G_1$ there are $O(G_1)$ choices for λa. Now if we consider $G_1 \leq g$ and $g^3abc \leq x$ we get

$$\begin{aligned}
abc &\leq x/g^3, \\
ab^2 &\leq x/g^3 \text{ since } c>b, \\
A(AJ)^2 &\leq x/G_1^3 \text{ since } A \leq a \leq 2A,\ B \leq b \leq 2B,\ AJ \leq B \leq 2AJ,\ G_1 \leq g, \\
A^3J^2 &= O(x/G_1^3), \\
A &= O\left(\frac{x^{1/3}}{G_1J^{2/3}}\right) \text{ and } B = O\left(\frac{x^{1/3}J^{1/3}}{G_1}\right).
\end{aligned}$$

Then since $B \leq b \leq 2B$ there are $O(x^{1/3}J^{1/3}/G_1)$ choices for b. Therefore to fix λ, a, b there are

$$O(G_1^{1+o(1)}(x^{1/3}J^{1/3}/G_1)) = O(x^{1/3+o(1)}J^{1/3}) = O(x^{1/3+o(1)}x^{1/42}) = O(x^{5/14+o(1)})$$

choices, since $J \leq x^{1/14}$. Once we fix λ, a, b then g, c have only $x^{o(1)}$ choices. Therefore to fix λ, a, b, g, c there are $O(x^{5/14+o(1)})$ choices.

We let the A, B, J run over powers of 2 and this introduces a factor of $x^{o(1)}$. Hence $N_3 = O(x^{5/14+o(1)})$. Hence $N = N_1+N_2+N_3 = O(x^{1/3})+O(x^{5/14+o(1)})+O(x^{5/14+o(1)}) = O(x^{5/14+o(1)})$.

Discussion. Our choices for parameters such as G were not arbitrary but optimal. We have used the optimal values for the parameters as this results in a shorter and clearer proof.

It would be best to make our bounds explicit and replace the $x^{o(1)}$ with a power of $\log x$. It is easy to see that these are two different problems. For the first problem we could use a result of Ramanujan [10] that states that there is an explicit constant K_α depending on α such that the number of divisors of n, $d(n) < K_\alpha n^\alpha$ for any positive number $0 < \alpha < 1$. For the second problem we need to consider the average of the divisor function over a polynomial on an interval. There are some results in this direction (see [6]), however, they depend on the coefficients of the polynomial in an unknown way.

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References

1. W.R. Alford, A. Granville, and C. Pomerance, *There are infinitely many Carmichael numbers*, Ann. of Math. **140** (1994), 703–722. MR **95k:**11114
2. W.R. Alford, A. Granville, and C. Pomerance, *On the difficulty of finding reliable witnesses*, Lecture Notes in Comput. Sci. **877** (1994), 1–16. MR **96d:**11136
3. I. Damgård, P. Landrock, and C. Pomerance, *Average case error estimates for the strong probable prime test*, Math. Comp. **61** (1993), 177–194. MR **94b:**11124
4. A. Granville, *Primality testing and Carmichael numbers*, Notices Amer. Math. Soc. **39** (1992), 696–700.
5. N. Koblitz, *A Course in Number Theory and Cryptography*, Springer Verlag, New York, 1987. MR **88i:**94001
6. Mohan Nair, *Multiplicative functions of polynomial values in short intervals*, Acta Arith. (3) **62** (1992), 257–269. MR **94b:**11093
7. R.G.E. Pinch, *The Carmichael numbers up to* 10^{15}, Math. Comp. **61** (1993), 381–392. MR **93m:**11137
8. C. Pomerance, *Carmichael numbers*, Nieuw Archief voor Wiskunde **11** (1993), 199–209. MR **94h:**11085
9. C. Pomerance, J.L. Selfridge, and S.S. Wagstaff, Jr., *The pseudoprimes to* $25 \cdot 10^9$, Math. Comp. **35** (1980), 1003-1026. MR **82g:**10030
10. S. Ramanujan, *Highly Composite Numbers*, Proc. London Math. Soc. (2) **14** (1915), 347–409.
11. P. Ribenboim, *The Book of Prime Number Records (Second Edition)*, Springer Verlag, New York, 1989. MR **90g:**11127

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